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X_{n-1}-forming sets of eigenvectors

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BY

ALBERT NIJENHUIS

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MATHEMATICS

X_{n-1}-FORMING SETS OF EIGENVECTORS

BY

ALBERT NIJENHUIS

(Communicated by Prof. J. A. SCHOUTEN at the meeting of March 31, 1951)

§ 1. *Introduction.* For a covariant tensor [1]) field of valence two in V_n with distinct eigenvalues one may seek conditions for the eigenvectors to be V_{n-1}-normal. This problem was solved by J. A. SCHOUTEN [2]), and necessary and sufficient conditions were given. These conditions, however, contain the eigenvectors of the tensor, so in order to determine whether a given tensor has this property one must first solve the characteristic equation. A. TONOLO solved the same problem for 3-space, first for R_3 [3]), later for V_3 [4]), and in both cases succeeded in giving these conditions a form containing neither the eigenvectors nor the eigenvalues of the tensor. SCHOUTEN gave the generalisation for V_n [5]) and showed that TONOLO's proofs could be abbreviated by more modern methods and notations.

It is possible to generalise the aforementioned problem to a mixed affinor field in X_n of valence two with distinct eigenvalues, and to seek conditions for the covariant eigenvectors to be X_{n-1}-forming. The metric and the connection of a V_n are then entirely superfluous. These conditions, again, will contain neither eigenvalues nor eigenvectors and they are in V_n shown to be equivalent to TONOLO–SCHOUTEN's criteria. The conditions will involve a hitherto unpublished differential comitant.

We will start from SCHOUTEN's considerations because they lead to the differential comitant.

We therefore take a V_n with coordinates $\xi^\varkappa$, $\varkappa, \lambda, \mu, \nu, \varrho, \sigma, \tau = 1, \ldots, n$, a metric tensor $g_{\mu\lambda}$, with its corresponding differential operator ∇_μ, and in this V_n we take a tensor field $h_{\mu\lambda}$.

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2) J. A. SCHOUTEN, Der Ricci-Kalkül, p. 196 f. (Berlin 1924).

3) ANGELO TONOLO, Sopra una classe di deformazione finite, Ann. mat. pur. appl. (4) 29, 99–114 (1949); id., Sulle equazioni di Weingarten relative ai sistemi tripli ortogonali di superficie isostatiche; Un. Roma e Ist. Naz. Alta Mat. Rend. Mat. e appl. (5) 2, 170–192 (1941).

4) ANGELO TONOLO, Sulle varietà riemanniane a tre dimensioni, Pont. accad. Sci. Acta 13, 29–53 (1949); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 438–444 (1949).

5) J. A. SCHOUTEN, Review on op. cit. 4), Math. Rev. 11, 461 (1950); id., Sur les tenseurs de V_n aux directions principales V_{n-1}-normales. Conférence au Colloque du Centre Belge de Recherches Mathématiques, 11–14 avril (1951).

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We shall assume that $h_{\mu\lambda}$ has distinct eigenvalues $\underset{1}{\lambda}, \ldots, \underset{n}{\lambda}$, and that none of them is zero. Let the unit eigenvectors be $\underset{1}{i^\varkappa}, \ldots, \underset{n}{i^\varkappa}$, and the anholonomic coordinate system they determine, (h), $h, i, j, k, l = 1, \ldots, n$ [6]. It is then a known fact that the $\underset{k}{i^\varkappa}$ are V_{n-1}-normal if and only if [2])

$$(1.\ 1) \qquad \underset{k}{i^\mu}\,\underset{l}{i^\lambda}\,\underset{j}{i^\varkappa}\,\nabla_\mu h_{\lambda\varkappa} = 0 \quad , \quad j, k, l \neq .$$

Now this condition is equivalent to the following set of equations:

$$(1.\ 2) \qquad \begin{cases} (a) & h_{[\nu}^{\cdot\cdot\varkappa}\,\nabla_\mu h_{\lambda]\varkappa} = 0, \\ (b) & \overset{-1}{h}{}_{[\nu}^{\cdot\cdot\varkappa}\,\nabla_\mu h_{\lambda]\lambda} = 0, \\ (c) & \overset{-1}{h}{}_{[\nu}^{\cdot\cdot\varkappa}\,\nabla_\mu \overset{-1}{h}{}_{\lambda]\varkappa} = 0, \end{cases}$$

where $\overset{-1}{h}{}^{\varkappa\lambda}$ is the inverse of $h_{\mu\lambda}$. This is the form SCHOUTEN gives. TONOLO used for $n = 3$ the minor [7]) of $h_{\mu\lambda}$ instead of $\overset{-1}{h}{}^{\varkappa\lambda}$, and then for this value of n the theorem also holds if $h_{\mu\lambda}$ has rank 2. For this detail, however, we refer to SCHOUTEN's publication.

SCHOUTEN proves as follows that equations (1. 2) are equivalent to (1. 1). Multiplying (1. 1) by $\underset{j}{\lambda}$ and using the fact that $\underset{j}{i^\varkappa}$ is an eigenvector, one obtains

$$(1.\ 3) \qquad \underset{k}{i^\mu}\,\underset{l}{i^\lambda}\,\underset{j}{i^\nu}\,h_{\nu}^{\cdot\cdot\varkappa}\,\nabla_\mu h_{\lambda\varkappa} = 0 \quad , \quad j, k, l \neq .$$

Alternation over j, k, l gives (1. 2a). The other equations are derived in a similar way. Hence (1. 2) is a set of necessary conditions.

Conversely, (1. 2) can be replaced by

$$(1.\ 4) \qquad \begin{cases} (a) & h_{[i}^{\cdot\cdot k}\,\nabla_j h_{h]k} = 0, \\ (b) & \overset{-1}{h}{}_{[i}^{\cdot\cdot k}\,\nabla_j h_{h]k} = 0, \\ (c) & \overset{-2}{h}{}_{[i}^{\cdot\cdot k}\,\overset{-1}{h}{}_{h}^{\cdot\cdot l}\,\nabla_{j]} h_{kl} = 0, \end{cases}$$

where $\overset{-2}{h}{}_{i}^{\cdot\cdot k}$ is an abbreviation for $\overset{-1}{h}{}_{i}^{\cdot\cdot l}\,\overset{-1}{h}{}_{l}^{\cdot\cdot k}$. (1. 4) can be replaced by

$$(1.\ 5) \qquad \begin{cases} (a) & (\underset{h}{\lambda} - \underset{i}{\lambda})\,\nabla_j h_{ih} + \text{cycl. } h, i, j = 0 \quad , \quad h, i, j \neq, \\ (b) & (\underset{h}{\lambda}{}^{-1} - \underset{i}{\lambda}{}^{-1})\,\nabla_j h_{ih} + \text{cycl. } h, i, j = 0 \quad , \quad h, i, j \neq, \\ (c) & (\underset{h}{\lambda}{}^{-2}\,\underset{i}{\lambda}{}^{-1} - \underset{i}{\lambda}{}^{-2}\,\underset{h}{\lambda}{}^{-1})\,\nabla_j h_{ih} + \text{cycl. } h, i, j = 0 \quad , \quad h, i, j \neq . \end{cases}$$

For every set of distinct h, i, j, (1. 5) form three homogeneous equations

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[6]) We remind that the summation convention for h, i, j, k is used *only* in case there are two equal indices, one covariant and one contravariant.

[7]) The *minor* of an affinor $h_{\lambda}^{\cdot\varkappa}$ is defined (cf. op cit. [5])) as the affinor density whose components are the minors of the corresponding elements of the transposed matrix of $h_{\lambda}^{\cdot\varkappa}$.

with three unknowns. The determinant of the set is $\neq 0$ because it is a product of nonvanishing factors. Hence (1. 2) leads to

$$\nabla_j h_{ih} = 0 \quad , \quad h, i, j \neq, \tag{1. 6}$$

which is equivalent to (1. 1). This proves the theorem.

§ 2. *The elimination of the metric.* The formula (1. 2a) is a starting point for the generalised problem. We will try to eliminate the metric from (1. 2a).

If an affinor field $h_{\lambda}^{\cdot\varkappa}$ is given (again the eigenvalues are assumed to be distinct), we can investigate when the covariant eigenvectors are X_{n-1}-forming. One way is to introduce an auxiliary metric tensor of rank n such that $h_{\mu}^{\cdot\varkappa} g_{\varkappa\lambda}$ is a tensor:

$$\begin{cases} (a) \quad g_{[\mu\lambda]} = 0, \\ (b) \quad h_{[\mu}^{\cdot\varkappa}\, g_{\lambda]\varkappa} = 0. \end{cases} \tag{2. 1}$$

With respect to (h) these equations become

$$\begin{cases} (a) \quad g_{ik} = g_{ki} \\ (b) \quad \underset{i}{\lambda}\, g_{ki} - \underset{k}{\lambda}\, g_{ik} = 0. \end{cases} \tag{2. 2}$$

Hence $g_{ik} = 0$ for $i \neq k$, and for each i ($i = 1, \ldots, $ n), g_{ii} is arbitrary but not zero. There are n^2 unknowns, $2\binom{n}{2} = n^2 - n$ equations, and n linearly independent solutions; so equations (2. 1) are independent.
Once a metric satisfying (2. 1) has been introduced, the eigenvectors of $h_{\lambda}^{\cdot\varkappa}$ are mutually perpendicular for this metric and the necessary and sufficient conditions are then (1. 2). The elimination of $g_{\mu\lambda}$ from (1. 2a) is now accomplished by making use of (2. 1) and of the equation

$$(\partial_\nu h_{[\mu}^{\cdot\varkappa})\, g_{\lambda]\varkappa} + h_{[\mu}^{\cdot\varkappa}\, \partial_{|\nu|}\, g_{\lambda]\varkappa} = 0, \tag{2. 3}$$

which is a consequence of (2. 1b). The left hand side of (1. 2a) becomes:

$$\begin{cases} h_{\varrho[\nu} \nabla_\mu h_{\lambda]}^{\cdot\varrho} = h_{\varrho[\nu}\, \partial_\mu h_{\lambda]}^{\cdot\varrho} + {}^1/_2\, h_{\varrho[\nu}\, h_{\lambda}^{\cdot\sigma}\, g^{\varrho\tau} (\partial_{\mu]}\, g_{\sigma\tau} + \partial_{|\sigma|}\, g_{\mu]\tau} - \partial_{|\tau|}\, g_{\mu]\sigma}) = \\ \quad = h_{\varrho[\nu}\, \partial_\mu h_{\lambda]}^{\cdot\varrho} + h_{[\nu}^{\cdot\tau}\, h_{\lambda}^{\cdot\sigma}\, \partial_{|\sigma|}\, g_{\mu]\tau} = \\ \quad = h_{[\nu}^{\cdot\tau} (\partial_\mu h_{\lambda]}^{\cdot\varrho})\, g_{\varrho\tau} - h_{[\lambda}^{\cdot\sigma} (\partial_{|\sigma|} h_{\nu}^{\cdot\tau})\, g_{\mu]\tau} = \\ \quad = h_{\varrho}^{\cdot\tau} (\partial_{[\mu} h_{\lambda}^{\cdot\varrho})\, g_{\nu]\tau} - h_{[\mu}^{\cdot\sigma} (\partial_{|\sigma|} h_{\lambda}^{\cdot\tau})\, g_{\nu]\tau} = \\ \quad = [h_{\varrho}^{\cdot\tau}\, \partial_{[\mu} h_{\lambda}^{\cdot\varrho} - h_{[\mu}^{\cdot\sigma}\, \partial_{|\sigma|}\, h_{\lambda}^{\cdot\tau}]\, g_{\nu]\tau}. \end{cases} \tag{2. 4}$$

Denoting the expression in brackets by $-{}^1/_2 H_{\mu\lambda}^{\cdot\cdot\tau}$, we arrive at the condition

$$H_{[\mu\lambda}^{\cdot\cdot\tau}\, g_{\nu]\tau} = 0. \tag{2. 5}$$

We may repeat the above argument for any solution of (2. 1); therefore (2. 5) should hold for every solution of (2. 1).
So, for a suitable choice of $\alpha_{\mu\lambda\nu}^{\cdot\cdot\cdot\varkappa\tau}$ and $\beta_{\mu\lambda\nu}^{\cdot\cdot\cdot\varkappa\sigma}$, alternating in all upper and in all lower indices, the equation

$$H_{[\mu\lambda}^{\cdot\cdot\tau}\, g_{\nu]\tau} = \alpha_{\mu\lambda\nu}^{\cdot\cdot\cdot\varkappa\tau}\, g_{\varkappa\tau} + \beta_{\mu\lambda\nu}^{\cdot\cdot\cdot\varkappa\sigma}\, h_{\sigma}^{\cdot\tau}\, g_{\varkappa\tau} \tag{2. 6}$$

should hold for *every* $g_{\varkappa\tau}$. Then, however, $g_{\varkappa\tau}$ can be omitted, and so the $g_{\varkappa\tau}$ is eliminated from (1. 2a):

$$H_{[\mu\lambda}^{\cdot\cdot\tau} A_{\nu]}^{\varkappa} = \alpha_{\mu\lambda\nu}^{\cdot\cdot\cdot\varkappa\tau} + \beta_{\mu\lambda\nu}^{\cdot\cdot\cdot\varkappa\sigma} h_{\sigma}^{\cdot\tau}. \tag{2. 7}$$

This condition is *necessary*. It will be proved in § 6 that (2. 7) is also *sufficient*. However, we will first turn to the generalised problem.

§ 3. *The quantity* $\overset{1,2}{H}{}_{\mu\lambda}^{\cdot\cdot\varkappa}$. We prove that the expression $H_{\mu\lambda}^{\cdot\cdot\varkappa}$ of § 2, defined by

$$H_{\mu\lambda}^{\cdot\cdot\varkappa} \overset{\text{def}}{=} 2\, h_{[\mu}^{\cdot\varrho}\, \partial_{|\varrho|}\, h_{\lambda]}^{\cdot\varkappa} - 2\, h_{\varrho}^{\cdot\varkappa}\, \partial_{[\mu}\, h_{\lambda]}^{\cdot\varrho}, \tag{3. 1}$$

is an affinor. The difference between (3. 1) and the same expression, written with ∇ instead of ∂, where ∇ belongs to any arbitrary symmetric connection ($\Gamma_{[\mu\lambda]}^{\varkappa} = 0$), is

$$\left\{\begin{aligned} & 2\, h_{[\mu}^{\cdot\varrho}\, \Gamma_{|\varrho|\lambda]}^{\tau}\, h_{\tau}^{\cdot\varkappa} - 2\, h_{[\mu}^{\cdot\varrho}\, \Gamma_{|\varrho\tau|}^{\varkappa}\, h_{\lambda]}^{\cdot\tau} + h_{\varrho}^{\cdot\varkappa}\, \Gamma_{[\mu|\tau|}^{\varrho}\, h_{\lambda]}^{\cdot\tau} = \\ & = 2\, h_{[\mu}^{\cdot\varrho}\, \Gamma_{|\varrho|\lambda]}^{\tau}\, h_{\tau}^{\cdot\varkappa} + 2\, h_{\mu}^{\cdot\varrho}\, \Gamma_{[\varrho\tau]}^{\varkappa}\, h_{\lambda}^{\cdot\tau} + 2\, h_{\tau}^{\cdot\varkappa}\, \Gamma_{\varrho[\mu}^{\tau}\, h_{\lambda]}^{\cdot\varrho} = 0. \end{aligned}\right. \tag{3. 2}$$

Since $H_{\mu\lambda}^{\cdot\cdot\varkappa}$ defined with ∇ instead of ∂ is an affinor and is shown to equal (3. 1), the affinor character of (3. 1) is proved.

Writing $h_{\lambda}^{\cdot\varkappa} = \underset{1}{h}{}_{\lambda}^{\cdot\varkappa} + \underset{2}{h}{}_{\lambda}^{\cdot\varkappa}$ and computing (3. 1) for this affinor, we find another affinor:

$$\boxed{\overset{1,2}{H}{}_{\mu\lambda}^{\cdot\cdot\varkappa} \overset{\text{def}}{=} \underset{1}{h}{}_{[\mu}^{\cdot\varrho}\, \partial_{|\varrho|}\, \underset{2}{h}{}_{\lambda]}^{\cdot\varkappa} + \underset{2}{h}{}_{[\mu}^{\cdot\varrho}\, \partial_{|\varrho|}\, \underset{1}{h}{}_{\lambda]}^{\cdot\varkappa} - \underset{1}{h}{}_{\varrho}^{\cdot\varkappa}\, \partial_{[\mu}\, \underset{2}{h}{}_{\lambda]}^{\cdot\varrho} - \underset{2}{h}{}_{\varrho}^{\cdot\varkappa}\, \partial_{[\mu}\, \underset{1}{h}{}_{\lambda]}^{\cdot\varrho}.} \tag{3. 3}$$

Putting

$$\underset{12}{h}{}_{\lambda}^{\cdot\varkappa} \overset{\text{def}}{=} \underset{1}{h}{}_{\lambda}^{\cdot\varrho}\, \underset{2}{h}{}_{\varrho}^{\cdot\varkappa} \tag{3. 4}$$

we have the following identity

$$\boxed{\overset{1,23}{H}{}_{\mu\lambda}^{\cdot\cdot\varkappa} + \overset{3,21}{H}{}_{\mu\lambda}^{\cdot\cdot\varkappa} = \overset{1,2}{H}{}_{\mu\lambda}^{\cdot\cdot\sigma}\, \underset{3}{h}{}_{\sigma}^{\cdot\varkappa} + \overset{3,2}{H}{}_{\mu\lambda}^{\cdot\cdot\sigma}\, \underset{1}{h}{}_{\sigma}^{\cdot\varkappa} - 2\, \underset{2}{h}{}_{[\mu}^{\cdot\varrho}\, \overset{1,3}{H}{}_{\lambda]\varrho}^{\cdot\cdot\varkappa}.} \tag{3. 5}$$

Introducing two arbitrary affinor fields $u^{\varkappa}$ and $v^{\varkappa}$, and the two fields

$${}'u^{\varkappa} = h_{\lambda}^{\cdot\varkappa}\, u^{\lambda} \quad , \quad {}'v^{\varkappa} = h_{\lambda}^{\cdot\varkappa}\, v^{\lambda} \tag{3. 6}$$

one can verify the identity

$$\boxed{\underset{'u}{\pounds}\, {}'v^{\varkappa} = u^{\nu}\, v^{\mu}\, H_{\nu\mu}^{\cdot\cdot\varkappa} + h_{\sigma}^{\cdot\varkappa}\, (\underset{u}{\pounds}\, {}'v^{\sigma} - \underset{v}{\pounds}\, {}'u^{\sigma}) - h_{\varrho}^{\cdot\sigma}\, h_{\sigma}^{\cdot\varkappa}\, \underset{u}{\pounds}\, v^{\varrho},} \tag{3. 7}$$

where $\underset{v}{\pounds}$ [8]) is the Lie-derivative [9]) with respect to $v^{\varkappa}$, for instance

$$\underset{v}{\pounds}\, u^{\varkappa} = v^{\mu}\, \partial_{\mu}\, u^{\varkappa} - u^{\mu}\, \partial_{\mu}\, v^{\varkappa} = -\underset{u}{\pounds}\, v^{\varkappa}. \tag{3. 8}$$

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[8]) J. A. SCHOUTEN (see op. cit. [5])) proposed the notation $\pounds$ instead of $\underset{L}{D}$ for the Lie-derivative because sometimes we need subscripts under the kernel, and all forms of the letter d already have one or more meanings.

[9]) J. A. SCHOUTEN und E. R. VAN KAMPEN, Beiträge zur Theorie der Deformation, Prac. Matematyczno-Fizycznych, Warszawa 41, 1 – 19 (1933); op. cit. [2]), p. 140 ff.; and p. 74 ff. respectively.

For a geometric interpretation, see also § 8.

The identities (3. 5) and (3. 7) play a central role in the solution of the generalised problem of TONOLO.

The expression

$$(3.\,9)\qquad \overset{1,2}{\Sigma}{}^{\varkappa}_{\mu\lambda} \overset{\text{def}}{=} \underset{1}{h}{}_{\mu}^{.\varrho}\, \partial_\varrho \underset{2}{h}{}_\lambda^{.\varkappa} - \underset{2}{h}{}_\lambda^{.\varrho}\, \partial_\varrho \underset{1}{h}{}_\mu^{.\varkappa} - \underset{1}{h}{}_\varrho^{.\varkappa}\, \partial_\mu \underset{2}{h}{}_\lambda^{.\varrho} + \underset{2}{h}{}_\varrho^{.\varkappa}\, \partial_\lambda \underset{1}{h}{}_\mu^{.\varrho}$$

is not an affinor, except when $\underset{1}{h}{}_\lambda^{.\varkappa}$ and $\underset{2}{h}{}_\lambda^{.\varkappa}$ commute: replacing ∂ by ∇ leads to adding the terms

$$(3.\,10)\qquad (\underset{1}{h}{}_\varrho^{.\varkappa}\, \underset{2}{h}{}_\tau^{.\varrho} - \underset{2}{h}{}_\varrho^{.\varkappa}\, \underset{1}{h}{}_\tau^{.\varrho})\, \Gamma^{\tau}_{\lambda\mu},$$

supposing ∇ belongs to a symmetric connection. This shows that we have the following differential comitants for a vector:

$$(3.\,11)\qquad \begin{cases} (\underset{1}{h}{}_\varrho^{.\varkappa}\, \underset{2}{h}{}_\lambda^{.\varrho} - \underset{2}{h}{}_\varrho^{.\varkappa}\, \underset{1}{h}{}_\lambda^{.\varrho})\, \partial_\mu v^\lambda + \overset{1,2}{\Sigma}{}^{\varkappa}_{\mu\lambda}\, v^\lambda, \\ \partial_\mu (\underset{1}{h}{}_\varrho^{.\varkappa}\, \underset{2}{h}{}_\lambda^{.\varrho} - \underset{2}{h}{}_\varrho^{.\varkappa}\, \underset{1}{h}{}_\lambda^{.\varrho})\, w_\varkappa - \overset{1,2}{\Sigma}{}^{\varkappa}_{\mu\lambda}\, w_\varkappa. \end{cases}$$

They can be generalised for any quantity. This proves that a pair of non-commuting mixed affinors of valence two and an arbitrary affinor have a first order differential comitant. — At present we shall not elaborate this further.

§ 4. *A. Tonolo's problem generalised for* X_n. Before investigating the conditions under which the covariant eigenvectors of $h_\lambda^{.\varkappa}$ (all eigenvalues are again supposed to be distinct) are X_{n-1}-forming, we recall that an equivalent problem is to determine when every pair of contravariant eigenvectors is X_2-forming. The equivalence is a consequence of the well-known fact that of a set of n linear homogeneous partial differential equations with one unknown variable, every set of $n-1$ of these equations is a *complete system* [10]) if and only if every *pair* of them is a complete system.

Consider now two arbitrary eigenvectors $u^\varkappa$ and $v^\varkappa$ of $h_\lambda^{.\varkappa}$, belonging to eigenvalues λ and μ, $\lambda \neq \mu$. We apply (3. 7) to $u^\varkappa$ and $v^\varkappa$. The left hand side becomes

$$(4.\,1)\qquad \underset{\lambda u}{\pounds}\, \mu v^\varkappa = (\underset{\lambda u}{\pounds}\, \mu)\, v^\varkappa + \mu \underset{\lambda u}{\pounds}\, v^\varkappa = \lambda v^\varkappa \underset{u}{\pounds}\, \mu - \mu \underset{v}{\pounds}\, \lambda u = \lambda v^\varkappa \underset{u}{\pounds}\, \mu - \mu u^\varkappa \underset{v}{\pounds}\, \lambda + \lambda\mu \underset{u}{\pounds}\, v^\varkappa,$$

and the second term of the right hand side

$$(4.\,2)\qquad \begin{cases} h_\sigma^{.\varkappa}\, (\underset{u}{\pounds}\, \mu v^\sigma - \underset{v}{\pounds}\, \lambda u^\sigma) = h_\sigma^{.\varkappa}\, (v^\sigma \underset{u}{\pounds}\, \mu + \mu \underset{u}{\pounds}\, v^\sigma - u^\sigma \underset{v}{\pounds}\, \lambda - \lambda \underset{v}{\pounds}\, u^\sigma) = \\ \qquad = \mu v^\varkappa \underset{u}{\pounds}\, \mu - \lambda u^\varkappa \underset{v}{\pounds}\, \lambda - (\lambda + \mu)\, h_\sigma^{.\varkappa} \underset{u}{\pounds}\, v^\sigma. \end{cases}$$

Thus (3. 7) takes the form

$$(4.\,3)\qquad u^\mu v^\lambda H_{\mu\lambda}^{..\varkappa} = (\lambda-\mu)\, u^\varkappa \underset{v}{\pounds}\, \lambda + (\lambda-\mu)\, v^\varkappa \underset{u}{\pounds}\, \mu + [h_\sigma^{.\varkappa} h_\varrho^{.\sigma} - (\lambda+\mu)\, h_\varrho^{.\varkappa} + \lambda\mu A_\varrho^\varkappa] \underset{u}{\pounds}\, v^\varrho.$$

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10) J. A. SCHOUTEN and W. v. D. KULK, Pfaff's Problem, p. 86 ff. (Oxford, 1949).

If now $u^\varkappa$ and $v^\varkappa$ are X_2-forming, the equations

$$(4.4) \qquad u^\mu \partial_\mu f = 0 \quad , \quad v^\mu \partial_\mu f = 0$$

form a *complete system*, i.e. $\underset{u}{£}\, v^\varkappa$ (cf. (3. 8)) is a linear combination of $u^\varkappa$ and $v^\varkappa$ [10]). In that case (4. 3) proves that $u^\mu v^\lambda H_{\mu\lambda}^{\cdot\cdot\varkappa}$ is also a linear combination of $u^\varkappa$ and $v^\varkappa$. Hence

$$(4.5) \qquad u^\mu\, v^\lambda\, H_{\mu\lambda}^{\cdot\cdot[\varkappa}\, u^\nu\, v^{\varrho]} = 0$$

is a *necessary* condition for any pair of eigenvectors $u^\varkappa$, $v^\varkappa$ to be X_2-forming.

If, conversely, (4. 5) holds for any pair of eigenvectors, then $T_\varrho^{\cdot\varkappa}\underset{u}{£}\, v^\varrho$, where $T_\varrho^{\cdot\varkappa}$ is the affinor in brackets in (4. 3), is also a linear combination of $u^\varkappa$ and $v^\varkappa$. Now for an arbitrary eigenvector $w^\varkappa$ belonging to the eigenvalue ν we find

$$(4.6) \qquad T_\varrho^{\cdot\varkappa}\, w^\varrho = \{\nu^2 - (\lambda + \mu)\,\nu + \lambda\mu\}\, w^\varkappa = (\nu - \mu)\,(\nu - \lambda)\, w^\varkappa.$$

Hence the only eigenvectors that are annihilated by $T_\varrho^{\cdot\varkappa}$ are $u^\varkappa$ and $v^\varkappa$. The consequence of this fact is that if $T_\varrho^{\cdot\varkappa}\underset{u}{£}v^\varrho$ has no component along any eigenvector distinct from $u^\varkappa$ and $v^\varkappa$, then $\underset{u}{£}v^\varkappa$ has this property, too. So, if (4. 5) holds, then $\underset{u}{£}v^\varkappa$ is a linear combination of $u^\varkappa$ and $v^\varkappa$. This proves that (4. 5) is also a *sufficient* condition for $u^\varkappa$ and $v^\varkappa$ to be X_2-forming.

With respect to the coordinate system (h), determined by the eigenvectors, (4. 5) can be written:

$$(4.7) \qquad H_{j\,i}^{\cdot\cdot h} = 0 \quad , \quad h, i, j \neq.$$

A third form is the following. Writing $\overset{2}{h}{}_\mu^{\cdot\varkappa}$ for $h_\mu^{\cdot\sigma}h_\sigma^{\cdot\varkappa}$, etc., we shall prove that the equation

$$(4.8) \qquad \boxed{H_{\mu\lambda}^{\cdot\cdot\varkappa} = \underset{0}{p}_{[\mu}\, A_{\lambda]}^\varkappa + \underset{1}{p}_{[\mu}\, h_{\lambda]}^{\cdot\varkappa} + \ldots + \underset{n-2}{p}_{[\mu}\, \overset{n-2}{h}{}_{\lambda]}^{\cdot\varkappa},}$$

with $\underset{0}{p}_\lambda, \ldots, \underset{n-2}{p}_\lambda$ suitably chosen, is equivalent to (4. 5). In order to prove this we write (4. 8) with respect to (h), then the only equations involving $\underset{0}{p}_1, \ldots, \underset{n-2}{p}_1$ are:

$$(4.9) \qquad \begin{cases} 2\, H_{1\,2}^{\cdot\cdot 2} = \underset{0}{p}_1 + \underset{2}{\lambda}\, \underset{1}{p}_1 + \underset{2}{\lambda}{}^2 \underset{2}{p}_1 + \ldots + \underset{2}{\lambda}{}^{n-2}\, \underset{n-2}{p}_1, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ 2\, H_{1\,n}^{\cdot\cdot n} = \underset{0}{p}_1 + \underset{n}{\lambda}\, \underset{1}{p}_1 + \underset{n}{\lambda}{}^2 \underset{2}{p}_1 + \ldots + \underset{n}{\lambda}{}^{n-2}\, \underset{n-2}{p}_1, \end{cases}$$

and these equations have exactly one solution because the determinant of the system is equal to $\prod_{1<i<k\leq n} (\underset{i}{\lambda} - \underset{k}{\lambda}) \neq 0$.

Similar arguments are valid for the other components of the p_λ. Because also, both sides of (4. 8) — written with respect to (h) — vanish when

h, i, j are not equal, (4. 8) is equivalent to (4. 5). (4. 8) contains neither eigenvectors nor eigenvalues.

§ 5. *The meaning of* $H_{\mu\lambda}^{\cdot\cdot\varkappa} = 0$. In case $H_{\mu\lambda}^{\cdot\cdot\varkappa} = 0$, equation (4. 8) is satisfied, the covariant eigenvectors of $h_{\lambda}^{\cdot\varkappa}$ are X_{n-1}-forming, and (4. 3) then reduces to

$$0 = (\lambda - \mu)\, u^{\varkappa} \underset{v}{\pounds}\, \lambda + (\lambda - \mu)\, v^{\varkappa} \underset{u}{\pounds}\, \mu. \qquad (5.\ 1)$$

This means that $\underset{v}{\pounds}\, \lambda = \underset{u}{\pounds}\, \mu = 0$. Thus we may conclude:

If $H_{\mu\lambda}^{\cdot\cdot\varkappa} = 0$, *every eigenvalue* $\underset{i}{\lambda}$ *is constant on every* X_{n-1}, *generated by its own covariant eigenvector* $\underset{i}{e_{\lambda}}$.

This statement implies that, if $\underset{1}{\lambda}, \ldots, \underset{n}{\lambda}$ are independent functions, they can be taken as coordinates. The X_{n-1}'s with $\underset{1}{\lambda} =$ const. are then automatically the X_{n-1}'s generated by the eigenvector $\underset{1}{e_{\lambda}}$, etc. In that case the characteristic equation

$$\det (h_{\mu}^{\cdot\varkappa} - \lambda A_{\mu}^{\varkappa}) = \underset{n}{\alpha} - \underset{n-1}{\alpha}\, \lambda + \ldots + (-\lambda)^n = 0 \qquad (5.\ 2)$$

must be solved as usual; however, *no integration* is necessary to find the X_{n-1}'s generated by the eigenvectors.

The $\underset{1}{\lambda}, \ldots, \underset{n}{\lambda}$ are independent functions if and only if

$$\det (\partial_{\mu} \underset{i}{\lambda}) \neq 0. \qquad (5.\ 3)$$

Because $\underset{1}{\lambda}, \ldots, \underset{n}{\lambda}$ are distinct, they are single-valued functions of $\underset{1}{\alpha}, \ldots, \underset{n}{\alpha}$. Consequently

$$\det (\partial \underset{p}{\alpha} / \partial \underset{k}{\lambda}) \neq 0 \quad , \quad p = 1, \ldots, n. \qquad (5.\ 4)$$

Conversely, if (5. 4) holds, $\underset{1}{\lambda}, \ldots, \underset{n}{\lambda}$ are distinct [11].

So we find as a necessary and sufficient condition for the independence of $\underset{1}{\lambda}, \ldots, \underset{n}{\lambda}$:

$$\det (\partial_{\mu} \underset{p}{\alpha}) \neq 0 \quad , \quad p = 1, \ldots, n. \qquad (5.\ 5)$$

Thus we have the following theorem:

If $H_{\mu\lambda}^{\cdot\cdot\varkappa} = 0$ *and* (5. 5) *holds, then for each* $i (i = 1, \ldots, n)$ *the* X_{n-1}*'s generated by the covariant eigenvector field* $\underset{i}{e_{\lambda}}$ *are the* X_{n-1}*'s of constant* $\underset{i}{\lambda}$.

[11]) The left hand side of (5. 4) equals the VANDERMONDE determimant of $\underset{1}{\lambda}, \ldots, \underset{n}{\lambda}$. This follows easily from the fact that

$$\partial \underset{p}{\alpha} / \partial \underset{i}{\lambda} = \underset{p-1}{\alpha} - \underset{i}{\lambda}\, \underset{p-2}{\alpha} + \underset{i}{\lambda}^2\, \underset{p-3}{\alpha} - \ldots + (-\underset{i}{\lambda})^{p-1}.$$

§ 6. *The relation to the formulae of Tonolo–Schouten.* For a metric space the formulae of TONOLO–SCHOUTEN (1. 2) were necessary and sufficient. In the absence of a metric we have (4. 8). The quantity $H_{\mu\lambda}^{\cdot\cdot\kappa}$ was found from (1. 2a), and we shall now investigate its relation to (1. 2b, c). Using again the notation $\overset{p}{h}{}_{\lambda}^{\cdot\kappa}$ for the p-th power of $h_{\lambda}^{\cdot\kappa}$, and denoting $\overset{1,2}{H}{}_{\mu\lambda}^{\cdot\cdot\kappa}$ by $\overset{p,q}{H}{}_{\mu\lambda}^{\cdot\cdot\kappa}$ for $\underset{1}{h}{}_{\lambda}^{\cdot\kappa} = \overset{p}{h}{}_{\lambda}^{\cdot\kappa}$ and $\underset{2}{h}{}_{\lambda}^{\cdot\kappa} = \overset{q}{h}{}_{\lambda}^{\cdot\kappa}$, we can write (1. 2) in the form (as one can prove by a computation analogous to (2. 4)):

$$(6.\ 1)\qquad \begin{cases} (a) & H_{[\mu\lambda}^{\cdot\cdot\tau}\, g_{\nu]\tau} = 0, \\ (b) & \overset{1,-1}{H}{}_{[\mu\lambda}^{\cdot\cdot\tau}\, g_{\nu]\tau} = 0, \\ (c) & \overset{-1,-1}{H}{}_{[\mu\lambda}^{\cdot\cdot\tau}\, g_{\nu]\tau} = 0. \end{cases}$$

Here $g_{\mu\lambda}$ satisfies (2. 1). The assumption is made that det $(h_{\lambda}^{\cdot\kappa}) \neq 0$. If this is not so, one must determine α so, that det $(h_{\lambda}^{\cdot\kappa} + \alpha A_{\lambda}^{\kappa}) \neq 0$, and replace $h_{\lambda}^{\cdot\kappa}$ by $h_{\lambda}^{\cdot\kappa} + \alpha\, A_{\lambda}^{\kappa}$.

We will now express $\overset{-1,1}{H}{}_{\mu\lambda}^{\cdot\cdot\kappa}$ and $\overset{-1,-1}{H}{}_{\mu\lambda}^{\cdot\cdot\kappa}$ in terms of $H_{\mu\lambda}^{\cdot\cdot\kappa}$. The identity (3. 5) can be written as follows for powers of $h_{\lambda}^{\cdot\kappa}$:

$$(6.\ 2)\qquad \overset{p,q+r}{H}{}_{\mu\lambda}^{\cdot\cdot\kappa} + \overset{r,q+p}{H}{}_{\mu\lambda}^{\cdot\cdot\kappa} = \overset{p,q}{H}{}_{\mu\lambda}^{\cdot\cdot\sigma}\, \overset{r}{h}{}_{\sigma}^{\cdot\kappa} + \overset{r,q}{H}{}_{\mu\lambda}^{\cdot\cdot\sigma}\, \overset{p}{h}{}_{\sigma}^{\cdot\kappa} - 2\, \overset{q}{h}{}_{[\mu}^{\cdot\sigma}\, \overset{p,r}{H}{}_{\lambda]\sigma}^{\cdot\cdot\kappa}.$$

Taking $p = r = 1$, we have

$$(6.\ 3)\qquad \overset{1,q+1}{H}{}_{\mu\lambda}^{\cdot\cdot\kappa} = \overset{1,q}{H}{}_{\mu\lambda}^{\cdot\cdot\sigma}\, h_{\sigma}^{\cdot\kappa} - \overset{q}{h}{}_{[\mu}^{\cdot\sigma}\, H_{\lambda]\sigma}^{\cdot\cdot\kappa}.$$

For $q = -1$, this gives:

$$(6.\ 4)\qquad \overset{1,-1}{H}{}_{\mu\lambda}^{\cdot\cdot\kappa} = \overset{-1}{h}{}_{[\mu}^{\cdot\varrho}\, H_{\lambda]\varrho}^{\cdot\cdot\sigma}\, \overset{-1}{h}{}_{\sigma}^{\cdot\kappa},$$

and from (6. 3) for $q = -2$ it follows that

$$(6.\ 5)\qquad \overset{1,-2}{H}{}_{\mu\lambda}^{\cdot\cdot\kappa} = \overset{1,-1}{H}{}_{\mu\lambda}^{\cdot\cdot\sigma}\, h_{\sigma}^{\cdot\kappa} - \overset{-2}{h}{}_{[\mu}^{\cdot\sigma}\, H_{\lambda]\sigma}^{\cdot\cdot\tau}\, \overset{-1}{h}{}_{\tau}^{\cdot\kappa},$$

while (6. 2) gives for $p = q = -1$, $r = +1$:

$$(6.\ 6)\qquad \overset{-1,-1}{H}{}_{\mu\lambda}^{\cdot\cdot\kappa} = \overset{1,-2}{H}{}_{\mu\lambda}^{\cdot\cdot\sigma}\, \overset{-1}{h}{}_{\sigma}^{\cdot\kappa} - \overset{-1,1}{H}{}_{\mu\lambda}^{\cdot\cdot\sigma}\, \overset{-2}{h}{}_{\sigma}^{\cdot\kappa} + 2\, \overset{-1}{h}{}_{[\mu}^{\cdot\sigma}\, \overset{-1,1}{H}{}_{\lambda]\sigma}^{\cdot\cdot\varrho}\, \overset{-1}{h}{}_{\varrho}^{\cdot\kappa}.$$

Substituting (6. 4, 5) in (6. 6) and simplifying, we obtain

$$(6.\ 7)\qquad \overset{-1,-1}{H}{}_{\mu\lambda}^{\cdot\cdot\kappa} = \overset{-1}{h}{}_{\mu}^{\cdot\varrho}\, \overset{-1}{h}{}_{\lambda}^{\cdot\sigma}\, H_{\varrho\sigma}^{\cdot\cdot\tau}\, \overset{-2}{h}{}_{\tau}^{\cdot\kappa}.$$

Hence (6. 1) has been reduced to

$$(6.\ 8)\qquad \begin{cases} (a) & H_{[\mu\lambda}^{\cdot\cdot\tau}\, g_{\nu]\tau} = 0, \\ (b) & \overset{-1}{h}{}_{\sigma}^{\cdot\tau}\, \overset{-1}{h}{}_{[\mu}^{\cdot\varrho}\, H_{\lambda|\varrho|}^{\cdot\cdot\sigma}\, g_{\nu]\tau} = 0, \\ (c) & \overset{-2}{h}{}_{\sigma}^{\cdot\tau}\, H_{\varrho\kappa}^{\cdot\cdot\sigma}\, \overset{-1}{h}{}_{[\mu}^{\cdot\varrho}\, \overset{-1}{h}{}_{\lambda}^{\cdot\kappa}\, g_{\nu]\tau} = 0. \end{cases}$$

As we shall prove now, this system is equivalent to (4. 8) — or to (4. 7).

Writing (6. 8) with respect to the coordinate system (h), we obtain:

$$(6.\,9)\quad \begin{cases} (a) & H_{ij}^{\cdot\cdot h}\, g_{hh} + \text{cycl. } h, i, j = 0 \quad , \quad h, i, j \neq, \\ (b) & H_{ij}^{\cdot\cdot h}\, (\underset{i}{\lambda}^{-1} + \underset{j}{\lambda}^{-1})\, \underset{h}{\lambda}^{-1}\, g_{hh} + \text{cycl. } h, i, j = 0 \quad , \quad h, i, j \neq, \\ (c) & H_{ij}^{\cdot\cdot h}\, \underset{i}{\lambda}^{-1}\, \underset{j}{\lambda}^{-1}\, \underset{h}{\lambda}^{-2}\, g_{hh} + \text{cycl. } h, i, j = 0 \quad , \quad h, i, j \neq. \end{cases}$$

Computation gives the value of the determinant of this system:

$$(6.\,10)\qquad g_{ii}\, g_{jj}\, g_{hh}\, \underset{i}{\lambda}^{-1}\, \underset{j}{\lambda}^{-1}\, \underset{h}{\lambda}^{-1}\, (\underset{i}{\lambda}^{-1} - \underset{j}{\lambda}^{-1})\, (\underset{j}{\lambda}^{-1} - \underset{h}{\lambda}^{-1})\, (\underset{h}{\lambda}^{-1} - \underset{i}{\lambda}^{-1}) \neq 0.$$

Hence (4. 7) follows from (6. 9). Because conversely, (6. 9) is a trivial consequence of (4. 7) the equivalence of the systems of formulae (4. 8) and (6. 9) — or (1. 2) — is proved.

§ 7. *Other forms of the condition for the metric and the non-metric case.* It is now clear that the solution of TONOLO's problem can be formulated without the introduction of a metric. Nevertheless, certain advantages accompany the metric, since in (4. 8) a number of unknowns occur, whereas in (1. 2) they do not. (1. 2), however, does contain the assumption that det. $(h_{\lambda}^{\cdot\varkappa}) \neq 0$. This superfluous condition is eliminated in the following alternative formulation. (6. 9a) offers the means of essentially replacing (6. 8) by (6. 8a). For each of the n linearly independent metrics (see § 2), (6. 8a) — or (6. 9a) — is a valid condition. These n equations are sufficient to imply $H_{ji}^{\cdot\cdot h} = 0$, i.e. (4. 7), since their determinant is necessarily non-vanishing because of the independence of the metrics. Indeed, even *three* metrics $\overset{1}{g}_{\mu\lambda}, \overset{2}{g}_{\mu\lambda}, \overset{3}{g}_{\mu\lambda}$ are sufficient, provided $\overset{[1}{g}_{hh}\, \overset{2}{g}_{ii}\, \overset{3]}{g}_{jj} \neq 0$ for *all* $h, i, j \neq$. For these three metrics one can take, for example,

$$(7.\,1)\qquad \begin{cases} (a) & \overset{1}{g}_{\mu\lambda} = g_{\mu\lambda}\,, \\ (b) & \overset{2}{g}_{\mu\lambda} = g_{\mu\lambda} + a\, h_{\mu\lambda} \quad , \quad a\, \underset{i}{\lambda} \neq -1, \\ (c) & \overset{3}{g}_{\mu\lambda} = g_{\mu\lambda} + \beta\, \overset{2}{h}_{\mu\lambda} \quad , \quad \beta\, \underset{i}{\lambda}^{2} \neq -1. \end{cases}$$

We thus obtain the equations

$$(7.\,2)\qquad \begin{cases} (a) & H_{[\mu\lambda}^{\cdot\cdot\tau}\, g_{\nu]\tau} = 0\,, \\ (b) & H_{[\mu\lambda}^{\cdot\cdot\tau}\, h_{\nu]\tau} = 0\,, \\ (c) & H_{[\mu\lambda}^{\cdot\cdot\tau}\, \overset{2}{h}_{\nu]\tau} = 0\,, \end{cases}$$

which no longer contain $\overset{-1}{h}_{\lambda}^{\cdot\varkappa}$ and remain valid whether or not det $(h_{\lambda}^{\cdot\varkappa}) = 0$. (7. 2) is equivalent to (6. 8); of course the equivalence is not between the individual equations of the two sets.

Now it is also obvious that the condition (2. 7) is *necessary and sufficient* in the non-metric case. It expresses the fact that for every solution of (2. 1), equation (1. 2a) is satisfied, or equivalently, that (6. 9a) is satisfied for any g_{hh}, g_{ii}, g_{jj}.

Because in (2. 7) a is merely to account for the part alternating in $\varkappa\tau$, (2. 7) is equivalent to

$$H_{[\mu\lambda}{}^{\cdot\cdot(\varkappa}A_{\nu]}^{\tau)} = \beta_{\mu\lambda\nu}{}^{\cdot\cdot\cdot\sigma(\varkappa}h_{\sigma}^{\cdot\tau)}. \tag{7. 3}$$

This condition also contains unknowns but that will probably be unavoidable if one wants to preserve formulas of elegant and useful size. The number of conditions to be imposed on $H_{\mu\lambda}^{\cdot\cdot\varkappa}$ is $3\binom{n}{3}$ in (1. 2) and $(n-2)\binom{n}{2} = 3\binom{n}{3}$ in (4. 7). One would expect, therefore, that these conditions might be expressed by means of a set of simple quantities — for example, $n-2$ bivectors or $n-2$ $(n-2)$-vectors or 3 trivectors, etc. — set equal to zero. Because, however, these bivectors, trivectors, etc. cannot be formed with the aid of $H_{\mu\lambda}^{\cdot\cdot\varkappa}$ and $h_{\lambda}^{\cdot\varkappa}$ alone, these seemingly most plausible expressions will probably not exist.

That a more complicated form does exist can be seen as follows. (7. 3) expresses that the $\varkappa\tau$-domain of $H_{[\mu\lambda}^{\cdot\cdot(\varkappa}A_{\nu]}^{\tau)}$ lies in the $\varkappa\tau$-domain of $A_{[\varrho}^{(\varkappa}h_{\sigma]}^{\cdot\tau)}$. Introducing collective indices ("Sammel-indizes")[12] A for $(\varkappa\tau)$ $(A = 1, \ldots, \binom{n+1}{2})$ and writing $H_{\mu\lambda\nu}^{\cdot\cdot\cdot A}$ for $H_{[\mu\lambda}^{\cdot\cdot(\varkappa}A_{\nu]}^{\tau)}$ and $P_{\varrho\sigma}^{\cdot\cdot A}$ for $A_{[\varrho}^{(\varkappa}h_{\sigma]}^{\cdot\tau)}$, we get the necessary and sufficient condition in a form not containing any unknowns:

$$H_{\mu\lambda\nu}^{\cdot\cdot\cdot[A}P_{\varrho_1\sigma_1}^{\cdot\cdot A_1}\ldots P_{\varrho_N\sigma_N}^{\cdot\cdot A_N]} = 0\,, \quad N = \binom{n}{2}. \tag{7. 4}$$

§ 8. *Some other remarks.* The identities (3. 5) and (3. 7) appear to play an essential part in the important stages of the discussion. On (3. 7) the derivation of the necessary and sufficient conditions depends, and on (3. 5) the proof of the equivalence of (4. 8) and the equations (1. 2) of TONOLO–SCHOUTEN. It may therefore be useful to indicate the geometrical ideas behind these hard-to-find identities. For (3. 7) the leading idea arose from considerations of analogous situations where similar — though simpler — identities hold.

A. The geometric meaning of the LIE-derivative of a contravariant vector in X_n can be made clear as follows: Consider the point transformation $\xi^\varkappa \rightarrow \xi^\varkappa + v^\varkappa dt$, applied to an infinitesimal vector $u^\varkappa dt'$. $u^\varkappa dt'$ represents two points of X_n : $\xi^\varkappa$ and $\xi^\varkappa + u^\varkappa dt'$. $\xi^\varkappa$ is now being transformed into $\xi^\varkappa + v^\varkappa dt$, and $\xi^\varkappa + u^\varkappa dt'$ into $\xi^\varkappa + u^\varkappa dt' + (v^\varkappa + dt'\, u^\mu \partial_\mu v^\varkappa)dt$. Hence the vector $u^\varkappa dt'$ displaced ("dragged along")[13] to $\xi^\varkappa + v^\varkappa dt'$ is $\overset{m}{u}{}^\varkappa(\xi^\varkappa + v^\varkappa dt)\,dt' = u^\varkappa dt' + u^\mu \partial_\mu v^\varkappa dt\, dt'$. The field value of $u^\varkappa dt'$ in $\xi^\varkappa + v^\varkappa dt$ is $(u^\varkappa + dt\, v^\mu \partial_\mu u^\varkappa)\,dt'$. Now the difference between those two vectors divided by dt is the LIE-derivative of $u^\varkappa dt'$ (cf. (3. 8) and fig. 1).

B. If one introduces in an X_n an anholonomic coordinate system (h) next to a holonomic one, $(\varkappa)$; the covariant measuring vectors $\overset{i}{e}_\lambda$ are not all gradients. The *object of anholonomity* is then introduced as follows

$$\Omega_{ji}^{h} \overset{\text{def}}{=} A_{ji}^{\mu\lambda}\,\partial_{[\mu}A_{\lambda]}^{h} \overset{*}{=} A_{ji}^{\mu\lambda}\,\partial_{[\mu}\overset{h}{e}_{\lambda]}, \tag{8. 1}$$

[12]) Cf. Einführung I p. 32 (see [1])).

[13]) See op. cit. [10]), p. 3; p. 140, and p. 75 respectively.

and is expressed in terms of the *covariant* measuring vectors of (h). It can also be expressed in terms of the *contravariant* measuring vectors; denoting by $\underset{j}{£}$ the Lie-derivative with respect to $\underset{j}{e}^{\varkappa}$, one can easily verify that

$$\underset{j}{£}\, \underset{i}{e}^{\varkappa} = -\, 2\, \underset{j}{e}^{k}\, \underset{i}{e}^{l}\, \Omega_{kl}^{h}\, A_{h}^{\varkappa} \overset{*}{=} -\, 2\, \Omega_{ji}^{h}\, A_{h}^{\varkappa}\,. \tag{8. 2}$$

The geometric significance is illustrated in fig. 2. At $\xi^{\varkappa}$ the vectors $\underset{i}{e}^{\varkappa}\, dt$ and $\underset{j}{e}^{\varkappa}\, dt$ are taken, and also at the points $\xi^{\varkappa} + \underset{i}{e}^{\varkappa}\, dt$, $\xi^{\varkappa} + \underset{j}{e}^{\varkappa}\, dt$. $\underset{j}{£}\, \underset{j}{e}^{\varkappa}\, dt^{2}$ is the "defect"-vector; it is a measure of the gaps in the "network".

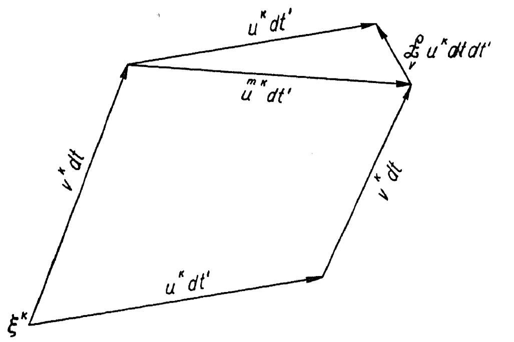

Fig. 1

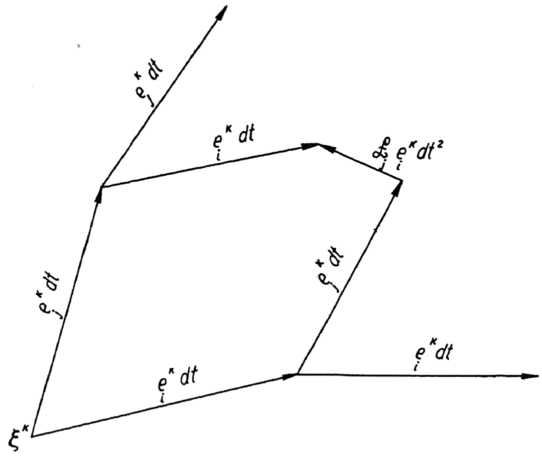

Fig. 2

C. In case an X_n is made into an L_n by the introduction of a linear connection $\Gamma_{\mu\lambda}^{\varkappa}$, the question of the geometrical significance of $S_{\mu\lambda}^{\cdot\cdot\varkappa} \overset{\text{def}}{=} \Gamma_{[\mu\lambda]}^{\varkappa}$ arises. This becomes clear from the identity

$$2\, u^{\mu}\, v^{\lambda}\, S_{\mu\lambda}^{\cdot\cdot\varkappa} = u^{\mu}\, \nabla_{\mu}\, v^{\varkappa} - v^{\mu}\, \nabla_{\mu}\, u^{\varkappa} - \underset{u}{£}\, v^{\varkappa}. \tag{8. 3}$$

If we choose the fields $u^{\varkappa}$ and $v^{\varkappa}$ so that the field values of $u^{\varkappa}$ and $v^{\varkappa}$ at $\xi^{\varkappa} + v^{\varkappa}\, dt$ and $\xi^{\varkappa} + u^{\varkappa}\, dt$ respectively, and the pseudo-parallel displaced values are equal, then we have in $\xi^{\varkappa}$:

$$u^{\mu}\, \nabla_{\mu}\, v^{\varkappa} = v^{\mu}\, \nabla_{\mu}\, u^{\varkappa} = 0, \tag{8. 4}$$

and we keep the formula

$$\underset{u}{£}\, v^{\varkappa} = -\, 2\, u^{\mu}\, v^{\lambda}\, S_{\mu\lambda}^{\cdot\cdot\varkappa}. \tag{8. 5}$$

It is a well-known fact that the right hand side of (8. 5) is a defect-vector. However, it seems that such defect-vectors arise in a natural way from using the Lie-derivative.

The general idea of a defect-vector is also of value in determining the geometric interpretation of the operator $\nabla_{[\nu}\nabla_{\mu]}$. One easily derives for L_n:

$$(v^{\nu}\nabla_{\nu}\, u^{\mu}\nabla_{\mu} - u^{\nu}\nabla_{\nu}\, v^{\mu}\nabla_{\mu})\, w_{\lambda} = -\, v^{\nu}\, u^{\mu}\, R_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa}\, w_{\varkappa} + (\underset{v}{£}\, u^{\mu})\, \nabla_{\mu}\, w_{\lambda}\,. \tag{8. 6}$$

If now the infinitesimal quadrangle of $u^\varkappa dt'$ and $v^\varkappa dt$ (fig. 1) is closed ($\underset{v}{\pounds}\, u^\varkappa = 0$), we get back the expression for the pseudo-parallel displacement around a surface element. If $u^\varkappa dt'$ is displaced parallel along $v^\varkappa dt$ and conversely, (8. 5) holds, and one gets back the formula for $\nabla_{[\nu}\nabla_{\mu]}$. This explains why the term with $S_{\nu\mu}^{\cdot\cdot\varkappa}$ occurs in that formula.

D. Consider the following facts:

1. In B and C the LIE-derivatives of vector fields with certain properties lead to a quantity — or object — of contravariant valence one and covariant valence two, alternating in these latter two indices.
2. The LIE-derivatives of vector fields are closely related to whether or not these fields are X_p-forming (cf. § 4).
3. $H_{\mu\lambda}^{\cdot\cdot\varkappa}$ arose from a problem concerning X_{n-1}-forming vector fields (cf. § 2).

Hence it was to be expected that $H_{\mu\lambda}^{\cdot\cdot\varkappa}$ would occur in a formula expressing in terms of $h_\lambda^{\cdot\varkappa}$ the LIE-derivatives of its eigenvectors with respect to one another. In (8. 5), moreover, such an expression is given for vectors displaced along one another. Since (8. 5) is a specialisation of (8. 3), and because $H_{\mu\lambda}^{\cdot\cdot\varkappa}$ is of degree two in $h_\lambda^{\cdot\varkappa}$ and its derivatives it was to be expected that there would be a general formula expressing the LIE-derivative of the transforms of vectors in terms of the LIE-derivative of the vectors themselves. This led to (3. 7). Specialisation for eigenvectors gave (4. 3).

The following considerations led to (3. 5). If the eigenvectors of $h_\lambda^{\cdot\varkappa}$ are X_{n-1}-forming, the same holds for every affinor $k_\lambda^{\cdot\varkappa}$ with distinct eigenvalues and the same eigenvectors. This must appear analytically by the possibility of expressing the quantity $K_{\mu\lambda}^{\cdot\cdot\varkappa}$ corresponding to (3. 1) in terms of $H_{\mu\lambda}^{\cdot\cdot\varkappa}$. Now $k_\lambda^{\cdot\varkappa}$ can be written as a polynomial of degree $n-1$ in $h_\lambda^{\cdot\varkappa}$; for if $\underset{1}{\varrho}, \ldots, \underset{n}{\varrho}$ are the eigenvalues of $k_\lambda^{\cdot\varkappa}$, the equation

$$(8.\ 7) \qquad k_\lambda^{\cdot\varkappa} = \underset{0}{\beta}\, A_\lambda^\varkappa + \underset{1}{\beta}\, h_\lambda^{\cdot\varkappa} + \ldots + \underset{n-1}{\beta}\, \overset{n-1}{h}{}_\lambda^{\cdot\varkappa}$$

leads to a set of linear equations with respect to (h):

$$(8.\ 8) \qquad \underset{i}{\varrho} = \underset{0}{\beta} + \underset{1}{\beta}\, \underset{i}{\lambda} + \underset{2}{\beta}\, \underset{i}{\lambda}{}^2 + \ldots + \underset{n-1}{\beta}\, \underset{i}{\lambda}{}^{n-1},$$

which can be solved. It must therefore be possible to express $\overset{p,q}{H}{}_{\mu\lambda}^{\cdot\cdot\varkappa}$ in terms of $H_{\mu\lambda}^{\cdot\cdot\varkappa}$. It seemed most plausible that a kind of a "product rule" as the rule of LEIBNITZ in differential calculus would exist. Now (3. 5) is not exactly that because *two* "product terms" arise in the left hand side. Still, (3. 5) is sufficient for the purpose since $K_{\mu\lambda}^{\cdot\cdot\varkappa}$ can be expressed in terms of $H_{\mu\lambda}^{\cdot\cdot\varkappa}$, $\partial_\mu \underset{0}{\beta}, \ldots, \partial_\mu \underset{n-1}{\beta}$ and powers of $h_\lambda^{\cdot\sigma}$ on account of the recurrence formulae (6. 3) and

$$(8.\ 9) \qquad \overset{p,q+1}{H}{}_{\mu\lambda}^{\cdot\cdot\varkappa} = \overset{p,q}{H}{}_{\mu\lambda}^{\cdot\cdot\tau}\, h_\tau^{\cdot\varkappa} - \overset{p+1,1}{H}{}_{\mu\lambda}^{\cdot\cdot\varkappa} + \overset{q,1}{H}{}_{\mu\lambda}^{\cdot\cdot\tau}\, \overset{p}{h}{}_\tau^{\cdot\varkappa} - 2\, \overset{q}{h}{}_{[\mu}^{\cdot\tau}\, \overset{p,1}{H}{}_{\lambda]\tau}^{\cdot\cdot\varkappa},$$

obtained from (6. 2) by taking $r = 1$.

For the check that $K_{ij}^{\cdot\cdot h} = 0$; $h, i, j \neq$, is a consequence of $H_{ij}^{\cdot\cdot h} = 0$; $h, i, j \neq$, we need not carry out these computations. If $u^{\varkappa}$ and $v^{\varkappa}$ are eigenvectors of $h_{\lambda}^{\cdot\varkappa}$ and $k_{\lambda}^{\cdot\varkappa}$ belonging to eigenvalues λ, μ and ϱ, σ respectively, we have the analogue of (4. 3):

$$(8.\ 10) \quad \left\{ \begin{aligned} u^{\mu} v^{\lambda} K_{\mu\lambda}^{\cdot\cdot\varkappa} = (\varrho-\sigma)\, u^{\varkappa} \underset{v}{£} \varrho + (\varrho-\sigma)\, v^{\varkappa} \underset{u}{£} \sigma + \\ + [k_{\sigma}^{\cdot\varkappa} k_{\varrho}^{\cdot\sigma} - (\varrho+\sigma)\, k_{\varrho}^{\cdot\varkappa} + \varrho\sigma A_{\varrho}^{\varkappa}] \underset{u}{£} v^{\varrho}. \end{aligned} \right.$$

Now, as a consequence of (4. 7), $\underset{u}{£} v^{\varrho}$ is a linear combination of $u^{\varkappa}$ and $v^{\varkappa}$. Hence by (8. 10) $K_{ij}^{\cdot\cdot h} = 0$; $h, i, j \neq$. The converse also holds provided $k_{\lambda}^{\cdot\varkappa}$ has distinct eigenvalues (read $k_{\lambda}^{\cdot\varkappa}$ for $h_{\lambda}^{\cdot\varkappa}$ and conversely in § 4 and in the above argument).